

The equation of state for the nuclear matter and the properties of the neutron star

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Abstract

The equation of state for the β stable nuclear matter is calculated numerically, and then the Tolman-Oppenheimer-Volkov(TOV) equation for the structure of the neutron star is solved in the fourth-order Runge-Kutta algorithm. It shows the mass and radius of the neutron star are functions of the central density of the neutron star and a maximum mass of 1.932 solar masses with a corresponding radius of 9.340km is obtained. Considering the equation of state of the nuclear matter must obey the causality, a new factor c is added in the nuclear potential energy formula. Therefore, with a new equation of state for the β stable nuclear matter when $c = 0.15$, a new maximum mass of 1.440 solar masses with a radius of 9.765km for the neutron star is obtained. Finally, the contribution of the cosmological constant to the structure of the neutron star is discussed, and we find the cosmological constant has minimal or negligible influence on the properties of the neutron star.

Key Words: Nuclear matter, Equations of state, Neutron stars

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Chang-Geng Liu is a diligent and excellent student. This article is based on his graduate thesis, which is chosen as one of the excellent theses in Beijing University of Technology in 2007.— Bao-Xi Sun

I. INTRODUCTION

Since the first pulsar was discovered in 1967, it has been generally believed that the pulsar is a rapidly rotating neutron star endowed with a strong magnetic field. Thereafter, studies upon the properties of neutron stars gained wide concerns.

Until now, many different views on the structure of the neutron star, especially the composition of the matter in the core of the neutron star, remain among theoretical physicists because it is difficult to analyze the components of the neutron star matter from the present astronomical observations[1]. The core of the neutron star, where the density is very large and thence the kaon condensation could happen, possibly consists of the strange hadronic matter including hyperons and there could also be the large-size quark matter in it. In 2006, it was concluded by Ozel that the condensates and unconfined quarks do not exist in the cores of neutron stars, because the mass and radius of the neutron star EXO 0748-676 rule out all the soft equations of state of the neutron star matter[2]. However, Alford et al. stated that the quark matter can be as stiff as the nuclear matter, and the corresponding hybrid or quark stars can reach a mass of 2.0 solar masses[3].

In the study of the structure of the neutron star, except the non-relativistic Brueckner theory[4, 5], the Quantum Hadrodynamics (QHD) based on the Walecka model has made great progress in the past 30 years[6, 7, 8, 9], which is also used to calculate the equation of state for the neutron star matter [10, 11]. To soften the equation of state for the neutron star matter, some research groups add self-interaction terms of mesons[12] in the Lagrangian density of QHD. At the meantime, considering the pair-correlation between nucleons, some scientists have established the relativistic Hartree-Bogliubov method[13, 14]. In addition, some others have developed the chiral hadronic model of QHD due to the chiral symmetry spontaneous breaking[15, 16].

In this article, the nuclear model in Refs. [17, 18] is utilized to calculate the equation of state for the neutron star matter and study further the properties of neutron stars although a similar work has been done by V. P. Psonis et al., where the relation between the asymmetric

energy of the nuclear matter and the properties of the neutron star is discussed within the same model[19]. The equilibrium conditions that the chemical potentials of protons, neutrons and electrons in the neutron star matter will be discussed in Section II. The equation of state for the neutron star matter and the requirement that the equation of state must meet to obey the causality will be discussed in Sections III and IV, respectively, and the numerical results on the properties of neutron stars are analyzed in Section V. The probable influence of the cosmological constant on the neutron star will be studied in Section VI. The summary is given in Section VII. It only needs some knowledge about quantum mechanics to calculate the equation of state with this model, so this work is suitable as a subject for the graduation thesis of an undergraduate physics major. Actually, this article is just based on Chang-Geng Liu's graduation thesis.

II. β EQUILIBRIUM CONDITIONS

Because a free neutron will undergo a beta weak decay process $n \rightarrow p + e^- + \bar{\nu}_e$, there are not only neutrons, but also a small fraction of protons and electrons in the neutron star matter. The neutron star is neutral. Hence, the density of the proton is equal to that of the electron, which means their Fermi momenta are the same as each other, $k_{f,p}=k_{f,e}$. When the neutron decay process takes place, the electron capture reaction $p + e^- \rightarrow n + \nu_e$ is going on. These two weak reactions will reach an equilibrium. This equilibrium can be expressed in terms of the chemical potentials for the three particle species

$$\mu_n = \mu_p + \mu_e, \quad (1)$$

where the chemical potential for a particle species can be expressed as

$$\mu_i = (k_{f,i}^2 c^2 + m_i^2 c^4)^{\frac{1}{2}}, \quad i = n, p, e \quad (2)$$

Thence, the Fermi momentum of the proton in the neutron star matter which reaches the beta equilibrium can be calculated as

$$k_{f,p} \approx \frac{k_{f,n}^2 + m_n^2 c^2 - m_p^2 c^2}{2(k_{f,n}^2 + m_n^2 c^2)^{\frac{1}{2}}}. \quad (3)$$

III. THE EQUATION OF STATE FOR THE NEUTRON STAR MATTER

Considering the influence of interactions between nucleons upon the equation of state, we mainly refer to Prakash's lectures on the asymmetric nuclear matter model to construct a simple nuclear potential energy formula[17, 18]. The saturation density of the symmetric nuclear matter $n_0 = 0.16 fm^{-3}$, and the binding energy per nucleon $BE = E/A - m_N c^2 = -16 MeV$, where m_N represents the mass of the nucleon, and k_0 is the nuclear compressibility that is applicable between 200 and 400MeV according to the experimental data and will be set to 360MeV in the discussion below. For the symmetric nuclear matter, the density of the neutron equals that of the proton, $n_n = n_p$, with the total nucleon density $n = n_n + n_p$. The average energy per nucleon for the symmetric nuclear matter will be modeled as [17, 18]

$$E(n, 0) = \frac{\varepsilon(n)}{n} = m_N c^2 + \frac{3}{5} \frac{k_f^2}{2m_N} + \frac{A}{2} u + \frac{B}{\sigma + 1} u^\sigma, \quad (4)$$

where $u = n/n_0$ and σ is dimensionless and A and B have the same dimension with the energy. The first term denotes the rest mass energy. The second term represents the average kinetic energy per nucleon with the corresponding Fermi momentum $k_f = (3\pi^2 \hbar^3 \frac{n}{2})^{\frac{1}{3}}$. At the saturation density, the average kinetic energy per nucleon is designated as $\langle E_f^0 \rangle = 22.1 MeV$, and then the second term can be rewritten as $\langle E_f^0 \rangle u^{\frac{2}{3}}$. The last two terms in Eq. (4) indicate the mean nuclear potential energy per nucleon. From the properties of the saturation nuclear matter, three constraint equations can be obtained to determine A, B and σ .

$$\langle E_f^0 \rangle + \frac{A}{2} + \frac{B}{\sigma + 1} = BE, \quad (5)$$

$$\frac{2}{3} \langle E_f^0 \rangle + \frac{A}{2} + \frac{\sigma B}{\sigma + 1} = 0, \quad (6)$$

$$\frac{10}{9} \langle E_f^0 \rangle + A + \sigma B = \frac{k_0}{9}, \quad (7)$$

which result in

$$\sigma = \frac{k_0 + 2 \langle E_f^0 \rangle}{3 \langle E_f^0 \rangle - 9BE}, \quad (8)$$

$$B = \frac{\sigma + 1}{\sigma - 1} \left[\frac{1}{3} \langle E_f^0 \rangle - BE \right], \quad (9)$$

$$A = BE - \frac{3}{5} \langle E_f^0 \rangle - B. \quad (10)$$

If k_0 is determined, A, B and σ are fixed accordingly. When $k_0 = 360 \text{ MeV}$, $\sigma = 1.922$, $A = -126.986 \text{ MeV}$ and $B = -74.098 \text{ MeV}$.

By the thermodynamic relation

$$p = n^2 \frac{d(\varepsilon/n)}{dn} = n \frac{d\varepsilon}{dn} - \varepsilon, \quad (11)$$

the pressure in the symmetric nuclear matter can be expressed as

$$p(n) = n_0 \left[\frac{2}{3} \langle E_f^0 \rangle u^{\frac{5}{3}} + \frac{A}{2} u^2 + \frac{\sigma B}{\sigma + 1} u^{\sigma+1} \right]. \quad (12)$$

As for the asymmetric nuclear matter, the densities of the proton and neutron are not the same any longer. Let us represent the proton and neutron densities in terms of a symmetry factor $\alpha = \frac{n_n - n_p}{n}$ with the proton density $n_p = \frac{1-\alpha}{2}n$ and the neutron density $n_n = \frac{1+\alpha}{2}n$.

As the asymmetric part of the mean energy per nucleon is approximately proportional to the square of α , it can be written as [20]

$$E(n, \alpha) = E(n, 0) + \alpha^2 \left[(2^{\frac{2}{3}} - 1) \langle E_f^0 \rangle (u^{\frac{2}{3}} - u) + S_0 u \right], \quad (13)$$

where $S_0 = 30 \text{ MeV}$ is the bulk symmetry energy parameter. Given the energy density $\varepsilon(n, \alpha) = n_0 u E(n, \alpha)$, the pressure can be expressed as

$$p(n, \alpha) = p(n, 0) + n_0 \alpha^2 \left[(2^{\frac{2}{3}} - 1) \langle E_f^0 \rangle \left(\frac{2}{3} u^{\frac{5}{3}} - u^2 \right) + S_0 u^2 \right]. \quad (14)$$

Neglecting the electro-magnetic interaction between the electron and the nucleon, the energy density and pressure of the neutron star matter can be expressed as

$$\varepsilon_{tot} = \varepsilon_e + \varepsilon(n, \alpha), \quad (15)$$

and

$$p_{tot} = p_e + p(n, \alpha), \quad (16)$$

respectively, where ε_e represents the contribution of the electron to the energy density and p_e to the pressure.

$$\varepsilon_e(k_f) = \varepsilon_0 \int_0^{k_{f,e}/m_e c} (u^2 + 1)^{\frac{1}{2}} u^2 du, \quad (17)$$

and

$$p_e(k_f) = n \frac{d\varepsilon_e}{dn} - \varepsilon_e = \varepsilon_0 \int_0^{k_{f,e}/m_e c} (u^2 + 1)^{-\frac{1}{2}} u^4 du \quad (18)$$

with $\varepsilon_0 = \frac{m_e^4 c^5}{\pi^2 \hbar^3}$.

α , n , $k_{f,p}$ and $k_{f,e}$ are all functions of $k_{f,n}$, so ε_{tot} and p_{tot} are both functions of $k_{f,n}$. Although the analytical expression between ε_{tot} and p_{tot} can not be deduced, the numerical form of the equation of state can be obtained.

The fourth-order Runge-Kutta algorithm is employed to solve the TOV equation for the structure of the neutron star.

$$\frac{dp(r)}{dr} = -\frac{GM(r)\varepsilon(r)}{c^2 r^2} \left[1 + \frac{p(r)}{\varepsilon(r)} \right] \left[1 + \frac{4\pi r^3 p(r)}{M(r)c^2} \right] \left[1 - \frac{GM(r)}{c^2 r} \right], \quad (19)$$

and

$$\frac{dM(r)}{dr} = \frac{4\pi r^2 \varepsilon(r)}{c^2}, \quad (20)$$

where G is the Newtonian gravitation constant, $\varepsilon(r)$ is the energy density at distance r and $p(r)$ the pressure, and $M(r)$ is the mass of the matter included in the sphere with the radius r . To analyze the dependence of the mass and radius versus the central pressure, a sequence of central pressures are taken into the TOV equation. The energy density corresponding to any pressure is calculated with the linear interpolation method, and the computer programmes are written with Fortran.

IV. AMENDMENT OF THE EQUATION OF STATE

The square of the rate of the speed of sound in the nuclear matter over that of light satisfies [18]

$$\left(\frac{c_s}{c}\right)^2 = \frac{B}{\rho c^2} = \frac{dp}{d\varepsilon} = \frac{dp/du}{d\varepsilon/du}. \quad (21)$$

By analysis from Eqs. (4) and (12), the speed of sound in the nuclear matter will exceed that of light at high nucleon density, i.e., the equation of state will violate the relativistic causality in this case. To make it obey the causality at the high nucleon density, the nuclear potential energy formula is revised as[18]

$$V(n, 0) = \frac{A}{2}u + \frac{B}{\sigma + 1} \frac{u^\sigma}{1 + cu^{\sigma-1}}, \quad (22)$$

where c is dimensionless and when it is set to zero, this formula returns to the unrevised case. Thereupon, the mean energy per nucleon for the symmetric matter turns into

$$\frac{\varepsilon(n)}{n} = m_N c^2 + \langle E_f^0 \rangle u^{\frac{2}{3}} + \frac{A}{2}u + \frac{B}{\sigma + 1} \frac{u^\sigma}{1 + cu^{\sigma-1}}, \quad (23)$$

and the pressure changes as

$$p(n) = n_0 \left(\frac{2}{3} < E_f^0 > u^{\frac{5}{3}} + \frac{A}{2} u^2 + \frac{B}{\sigma + 1} \frac{\sigma u^{\sigma+1} + c u^{2\sigma}}{(1 + c u^{\sigma-1})^2} \right). \quad (24)$$

The parameters A , B , and σ are fixed in the same way as in Section III, but another parameter c has to be determined firstly. From our calculations, c is suitable between 0.1 and 0.3. We set $c=0.15$ and obtain $\sigma=2.246$, $A = -119.468 \text{ MeV}$ and $B = -80.562 \text{ MeV}$ for the nuclear compressibility $k_0 = 360 \text{ MeV}$.

Fig. 1 shows the dependence of the energy density versus the nucleon density of the neutron star matter before and after amendment. At the low nucleon density, the curves fit together, while at the high nucleon density, they deviate from each other obviously. As can be seen in Fig. 1, the equation of state becomes softer after amendment. As is shown in Fig. 2, the curve of the equation of state after amendment only approaches the relativistic limit $p = \frac{1}{3}\varepsilon$ and stays distinctly under the causal limit $p = \varepsilon$, which indicates the equation of state after amendment obeys the causality at the high nucleon density.

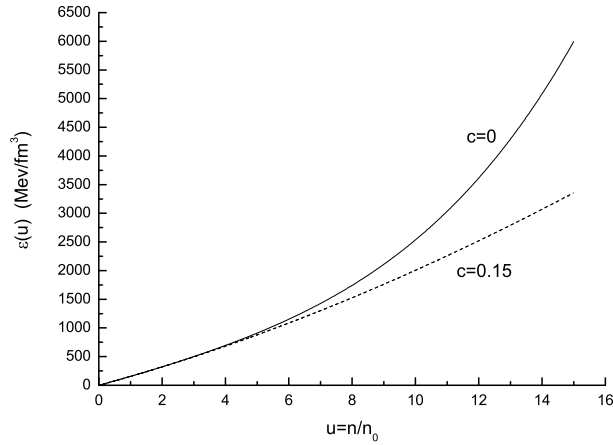


Fig. 1 The energy density of the β stable nuclear matter versus the nucleon density before and after amendment.

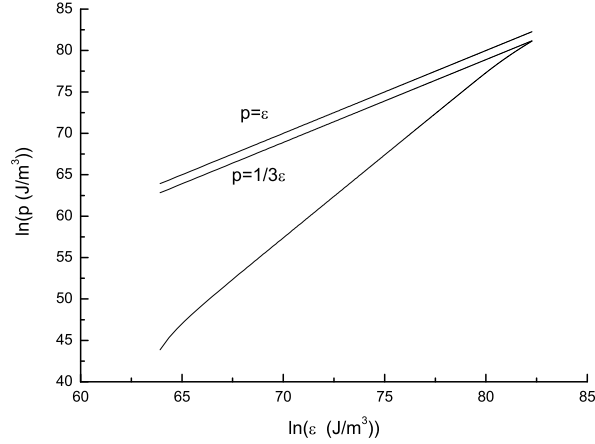


Fig. 2 The equation of state for the β stable nuclear matter after amendment

V. THE PROPERTIES OF NEUTRON STARS

By solving the TOV equation, the masses and radii of neutron stars under different central densities are obtained. In Fig. 3, the dashed curve represents the dependence of the radius versus the central density of neutron stars when $c = 0.15$ and the solid one represents that when $c = 0$. Other than minimal deviation at the high nucleon density, the two curves fit together well basically, so the revised equation of state has little influence on the radius of the neutron star. However, the amendment of the equation of state does make great difference to the relation between the mass and the central density of neutron stars, as can be seen in Fig. 4, where the dashed curve is the case of $c = 0.15$, and the solid one is the case of $c = 0$. The maximum point of the solid curve corresponds to a maximum mass $M_{max} = 1.932M_{sun}$ with a radius $R = 9.370km$, where M_{sun} is the solar mass, while the one of the dashed curve corresponds to $M_{max}=1.440M_{sun}$ and $R=9.765km$.

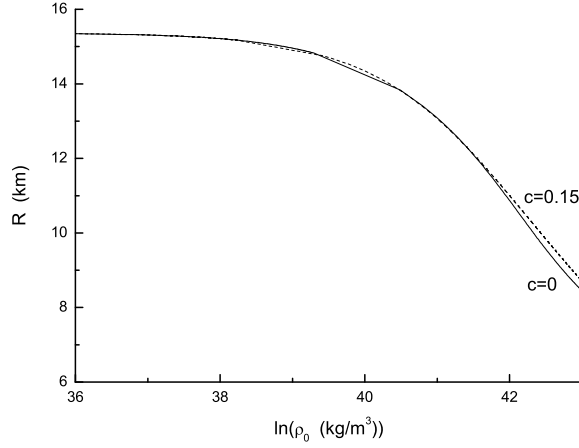


Fig. 3 The radius versus the central density of neutron stars before and after amendment.

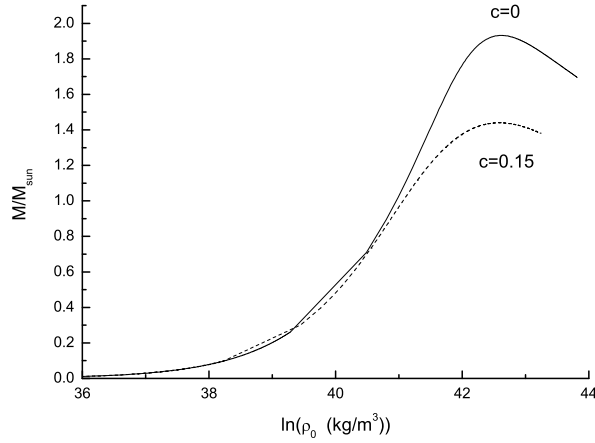


Fig. 4 The mass versus the central density of neutron stars before and after amendment.

VI. THE INFLUENCE OF THE COSMOLOGICAL CONSTANT UPON THE PROPERTIES OF THE NEUTRON STAR

If the cosmological constant Λ is positive, it has an upper limit of $4 \times 10^{-35} s^{-2}$, and if negative, it has a lower limit of $-2 \times 10^{-35} s^{-2}$ [21]. Considering the possible influence of Λ on the structure of the neutron star, the factor $[1 + \frac{4\pi r^3 p(r)}{M(r)c^2}]$ in the TOV equation in Eq. (19) should be changed into $[1 + \frac{4\pi r^3 p(r)}{M(r)c^2} - \frac{\Lambda r^3}{2GM(r)}]$.

We take a series of values of Λ between the upper and lower limits into the revised TOV equation and solve it using the equation of state before the causality amendment. The

maximum masses in the unit of the solar mass and the corresponding radii in the unit of the kilometer under different values of Λ remain unchanged until the third place behind the decimal point. It implies the cosmological constant Λ has only minimal or even negligible influence on the properties of the neutron star.

VII. SUMMARY

The equation of state for the neutron star matter is calculated numerically, with the β equilibrium conditions considered and the contribution of the electron included, and then the TOV equation for the structure of the neutron star is solved. It shows the mass and radius of the neutron star are functions of the central density. Considering the equation of state of the nuclear matter must obey the causality, a new factor c is added in the nuclear potential energy formula. Therefore, with a new equation of state of the neutron star matter when $c = 0.15$, a maximum mass and the corresponding radius of the neutron star are obtained. Finally, the contribution of the cosmological constant to the structure of the neutron star is discussed, and we find the cosmological constant has minimal or negligible influence on the properties of the neutron star.

In the course of working on the graduation thesis on this topic, the student can learn the knowledge about the degenerate Fermi gas in quantum mechanics, know about the related contents of the special and general relativity and has an initial recognition of the structure of the neutron star. Besides, Chang-Geng Liu has learned to skillfully write computer programmes with Fortran to solve physical problems, utilize Origin software to draw figures and edit scientific articles with Latex. In addition, he has made great progress in writing and translating scientific articles.

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